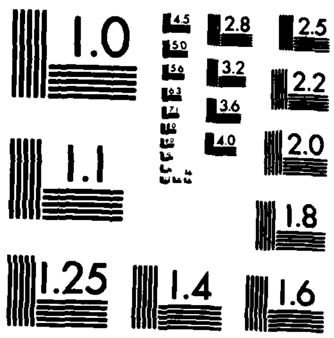
AD-A127 924 EXPONENTIAL APPROXIMATIONS FOR TWO CLASSES OF AGING DISTRIBUTIONS(U) CITY COLL NEW YORK DEPT OF MATHEMATICS M BROWN ET AL. FEB 83 CUNY-MB2 AFOSR-TR-83-0318 1/1

UNCLASSIFIED AFOSR-82-0024 F/G 12/1 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AFOSR-TR- 83-0318

(12)

AD A127924

Exponential Approximations

for Two Classes of Aging Distributions

by Mark Brown
The City College, CUNY

and Guangping Ge
CUNY Graduate Center

February, 1983

City College, CUNY Report No. MB2
AFOSR Technical Report No. 82-02
AFOSR Grant No. 82-0024

83 05 06 -195

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 83-0318	2. GOVT ACCESSION NO. AD-A127924	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) EXPONENTIAL APPROXIMATIONS FOR TWO CLASSES OF AGING DISTRIBUTIONS		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL
		6. PERFORMING ORG. REPORT NUMBER CUNY No. MB2
7. AUTHOR(s) Mark Brown and Guangping Ge		8. CONTRACT OR GRANT NUMBER(s) AFOSR-82-0024
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Mathematics The City College, CUNY New York NY 10031		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS PE61102F; 2304/A6
11. CONTROLLING OFFICE NAME AND ADDRESS Mathematical & Information Sciences Directorate Air Force Office of Scientific Research Bolling AFB DC 20332		12. REPORT DATE FEB 83
		13. NUMBER OF PAGES 14
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abst ..t entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

NBUE and NWUE distributions; exponential approximations; inequalities.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Inequalities are derived for the quality of exponential approximation to NBUE (new better than used) and NWUE (new worse than used) distributions.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Summary. Inequalities are derived for the quality of exponential approximation to NBUE (new better than used) and NWUE (new worse than used) distributions.

1. Introduction. If a random variable is exponentially distributed with $\mu = EX$ and $\mu_2 = EX^2$, then $\mu_2 = 2\mu^2$. Defining $\rho = |\frac{\mu_2}{2\mu^2} - 1|$, it is tempting to conjecture that under mild restrictions a distibution with small ρ is approximately exponential. That restrictions are needed is seen by the example, $\Pr(X=0) = \Pr(X=1) = \frac{1}{2}$, for which $\rho = 0$.

The scale invariant quantity, ρ, was suggested by Keilson [7]. It has an interesting interpretation. Define $G(x) = \mu^{-1} \int_0^x \bar{F}(s)ds$, the stationary renewal distribution corresponding to F. Then $\mu_G = \mu_2/2\mu$ and $\rho = |\frac{\mu_G}{\mu} - 1|$. The parameter ρ is thus the scaled (by μ) distance between μ and μ_G. For F exponential, $F = G$ and thus $\mu = \mu_G$.

The problem of interest can be stated as follows: Given a class $\mathcal{S}$ of distributions, along with the first two moments μ and μ_2 find upper bounds for $\sup_{t,F\in\mathcal{S}} |\bar{F}(t)-e^{-t/\mu}|$ in terms of ρ.

The above problem for the class of completely monotone distributions (mixtures of exponential distributions) was studied by Keilson [7], Heyde [5], Heyde and Leslie [6], Hall [4], and Brown [1].

Brown [1] considered the class of IMRL (increasing mean residual life) distributions on $[0,\infty)$ deriving:

(1.1) $$\sup_t |\bar{F}(t)-e^{-t/\mu}| \leq \frac{\rho}{\rho+1}$$

(1.2) $$\sup_{B\in\mathcal{B}} |F(B)-G(B)| \leq \frac{\rho}{\rho+1}$$

(1.3) $$\sup_{B\in\mathcal{B}} |G(B) - \int_B \mu^{-1}e^{-t/\mu}dt| \leq \frac{\rho}{\rho+1}$$

(1.4) $$\sup_t |\bar{G}(t)-e^{-t/\mu_G}| \leq \frac{\rho}{\rho+1} .$$

In (1.2) and (1.3) above β is the collection of Borel subsets of $[0,\infty)$. The quantity $\frac{\rho}{\rho+1}$ was shown to be the best upper bound for (1.1) and (1.2) even within the subclass of completely monotone distributions.

Brown [2] considered the class of IFR (increasing failure rate) distributions. It turns out that in this case (1.1) and (1.2) hold with $\frac{\rho}{\rho+1}$ replaced by 2ρ, and (1.3) and (1.4) hold with $\frac{\rho}{\rho+1}$ replaced by ρ. The bound 2ρ is the best bound for (1.1), among bounds of the form $c\rho^{\alpha}$.

In this paper we consider the problem for F NBUE (new better than used in expectation) and for F NWUE (new worse than used in expectation). These are the weakest among the commonly studied classes of aging distributions, and it is often easy to demonstrate that a distribution belongs to one of these classes (NBUE and NWUE are defined in Section 2). The methods of Brown ([1], [2]) do not generalize to these cases because the partial ordering between F and G is too weak. Instead we use Fourier methods adopted from Feller [3]. Our main result is that for F NBUE or NWUE:

$$\sup_t |\overline{F}(t)-e^{-t/\mu}| \leq A\rho^{1/2} \tag{1.5}$$

where $A = \frac{4\sqrt{6}}{\pi} \approx 3.119$. For the NBUE case we show that the best bound of the form $c\rho^{\alpha}$ has $\alpha = 1/2$ and $1 \leq c \leq \frac{4\sqrt{6}}{\pi}$. Thus the potential improvement in (1.5) for F NBUE is the lowering of the constant from 3.119 to 1. This remains true even within the subclass of IFRA distributions.

2. Definitions and Preliminary Results. A distribution F on $[0,\infty)$ with $F(0) < 1$ and finite mean μ is defined to be NBUE if $E(X-t|X>t) \leq \mu$ for all $t \geq 0$ with $\overline{F}(t) > 0$. Since $E(X-t|X>t) = \mu\overline{G}(t)/\overline{F}(t)$, it follows that F is NBUE if and only if F is stochastically larger than G, the stationary renewal distribution corresponding to F. Define h_G to be the failure rate function of G and note that $h_G(t) = [E(X-t|X>t)]^{-1}$, thus F is NBUE if and only if $h_G(t) \geq \mu^{-1}$ for all $t \geq 0$ with $\overline{F}(t) > 0$.

A distribution F on $[0,\infty)$ with $F(0) > 1$ and finite mean is defined to be NWUE if $E(X-t|X>t) \geq \mu$ for all $t \geq 0$ with $\overline{F}(t) > 0$. This is equivalent to F being stochastically smaller than G, and also to $h_G \leq \mu^{-1}$.

Lemma 2.1. If F is NBUE then $\overline{G}(t) \leq e^{-t/\mu}$ for all $t \geq 0$; for F NWUE, $\overline{G}(t) \geq e^{-t/\mu}$ for all $t \geq 0$.

Proof. For F NBUE let t_0 be the smallest number such that $\overline{F}(t_0) = 0$, with $t_0 = \infty$ if $\overline{F}(t) > 0$ for all t. Now $h_G(t) \geq \mu^{-1}$ for $0 \leq t < t_0$, thus $\overline{G}(t) \leq e^{-t/\mu}$ for $0 \leq t < t_0$. If $t_0 < \infty$ then for $t > t_0$ $\overline{G}(t) = 0 \leq e^{-t/\mu}$. If F is NWUE then $\overline{F}(t) > 0$ for all t, for if $\overline{F}(t_0) = 0$ for a finite t_0 then $\lim_{t \to t_0} E(X-t|X>t) = 0 < \mu$. Thus $h_G(t) \leq \mu^{-1}$ for all $t \geq 0$ and $\overline{G}(t) \geq e^{-t/\mu}$ for all $t \geq 0$.

The following inequality (Lemma 2.2) is quite an important tool in deriving our subsequent results. It relies heavily on a smoothing result of Feller [3] (Lemma 1. p. 510).

Lemma 2.2. Let F_1, F_2 be probability distributions on $[0,\infty)$ with finite means μ_1 and μ_2. Assume that F_1 is either stochastically larger or smaller than F_2, and that F_2 is differentiable with $F_2'(x) \leq \mu_1^{-1}$ for all $x \geq 0$. Then:

$$\sup_x |F_1(x)-F_2(x)| \leq A[|\mu_1-\mu_2|/\mu_1]^{1/2}$$

where $A = 4\sqrt{6}/\pi$.

Proof. By Feller [3], Lemma 1 p. 510,

$$\sup|F_1(x)-F_2(x)| \leq 2 \sup_t |T_{\Delta(t)}| + \frac{24}{\pi\mu_1 T} \tag{2.3}$$

where $\Delta(x) = F_1(x)-F_2(x)$

$$T_{\Delta(t)} = \int_{-\infty}^{\infty} \Delta(t-x)V_T(x)dx$$

$$V_T(x) = \frac{1-\cos Tx}{\pi Tx^2}.$$

Now, assume that F_1 is stochastically larger than F_2. Then:

$$|T_{\Delta(t)}| = \left|\int_{-\infty}^{\infty} [F_1(t-x)-F_2(t-x)] \frac{1-\cos Tx}{\pi Tx^2} dx\right|$$

$$= \int_{-\infty}^{t} [\bar{F}_1(t-x)-\bar{F}_2(t-x)] \frac{1-\cos Tx}{\pi Tx^2} dx$$

$$\leq \frac{T}{2\pi} \int_{-\infty}^{t} [\bar{F}_1(t-x)-\bar{F}_2(t-x)]dx$$

$$= \frac{T}{2\pi} (\mu_1-u_2).$$

Thus from (2.3):

$$\sup|F_1(x)-F_2(x)| \le \frac{1}{\pi} [T(\mu_1-\mu_2) + \frac{24}{\mu_1 T}] .$$

Define $L(T) = T(\mu_1-\mu_2) + \frac{24}{\mu_1 T}$, then a routine differentiation argument gives:

$$\min_{T>0} L(T) = L[(24/T\mu_1(\mu_2-\mu_1))^{1/2}] = 4\sqrt{6}\,[1-(\mu_2/\mu_1)]^{1/2}$$

and the result is proved.

If F_2 is stochastically larger than F_1 the analogous result follows by similar argument.

3. NBUE Results. Assume that F is NBUE. Recall that $\overline{G}(t) \le \overline{F}(t)$ and $G(t) \le e^{-t/\mu}$ for all $t \ge 0$, where G is the stationary renewal distribution corresponding to F. Note that $G'(x) = \overline{F}(x)/\mu \le \mu^{-1}$ for all x. Applying lemma 2.2 with $F_1 = F$, $F_2 = G$ we obtain:

$$(3.1) \qquad \sup|\overline{F}(x)-\overline{G}(x)| \le \frac{4\sqrt{6}}{\pi} (1 - \frac{\mu_G}{\mu}) = \frac{4\sqrt{6}}{\pi} \rho^{1/2} .$$

By Brown [1] remark 4.14, for F NBUE:

$$(3.2) \qquad \sup|\overline{G}(t)-e^{-t/\mu}| \le \sup|G(B) - \int_B \mu^{-1} e^{-t/\mu} dt| \le \rho .$$

Since F NBUE implies $\overline{G}(x) \le \min(\overline{F}(x), e^{-x/\mu})$ for all $x \ge 0$, it follows that:

$$(3.3) \qquad \sup|\bar{F}(x)-e^{-x/\mu}| \le \max(\sup|\bar{F}(x)-\bar{G}(x)|, \sup|\bar{G}(x)-e^{-x/\mu}|) \le \frac{4\sqrt{6}}{\pi}\rho^{1/2} .$$

Next, by simple computation:

$$(3.4) \qquad \sup|e^{-t/\mu}-e^{-t/\mu_G}| \le 1-(\mu_G/\mu) = \rho .$$

Moreover, $e^{-t/\mu} \ge \max(\bar{G}(t), e^{-t/\mu_G})$, thus:

$$(3.5) \qquad \sup|\bar{G}(t)-e^{-t/\mu_G}| \le \max(\sup|\bar{G}(t)-e^{-t/\mu}|, \sup|e^{-t/\mu}-e^{-t/\mu_G}|) \le \rho .$$

We summarize these results in Theorem 3.6.

Theorem 3.6. Let F be NBUE. Then:

$$\sup|\bar{F}(x)-e^{-x/\mu}| \le A\rho^{1/2}$$

$$\sup|\bar{F}(x)-\bar{G}(x)| \le A\rho^{1/2}$$

$$\sup|\bar{G}(x)-e^{-x/\mu}| \le \sup|G(B)- \int_B \mu^{-1}e^{-t/\mu}dt| \le \rho$$

$$\sup|\bar{G}(x)-e^{-x/\mu_G}| \le \rho$$

where $A = \frac{4\sqrt{6}}{\pi}$ and $\rho = 1-(\mu_2/2\mu^2)$.

Corollary (3.7) below presents a limit theorem for NBUE distributions.

Corollary 3.7. Let $\{X_n, n \ge 1\}$ be a sequence of NBUE random variables with $\mu_n = EX_n$, $\mu_{2,n} = EX_n^2$ and $\rho_n = 1-(\mu_{2n}/2\mu_n^2)$. Then X_n/μ_n converges in distribution to an exponential distribution if and only if

$\lim \rho_n = 0$, in which case the mean of the limiting exponential distribution equals 1.

Proof. The sufficiency of the condition $\lim \rho_n = 0$ follows from Theorem 3.6. To prove necessity assume that $\lim \Pr(X_n > t\mu_n) = e^{-ct}$ for all $t \geq 0$, and some $c > 0$. Let G_n denote the stationary renewal distribution corresponding to X_n, and H_n the stationary renewal distribution corresponding to X_n/μ_n. Then $\overline{H}_n(t) = \overline{G}_n(t\mu_n)$ and $\rho_n = 1 - \int_0^\infty \overline{H}_n(t)dt$.

Now:

$$\lim_{n\to\infty} \overline{H}_n(t) = 1-\lim_{n\to\infty} H_n(t) = 1-\lim_{n\to\infty} \int_0^t \Pr(X_n > s\mu_n)ds = 1-[(1-e^{-ct})/c] .$$

Since X_n is NBUE, so is X_n/μ_n, and it thus follows from Lemma 2.1 that:

$$\overline{H}_n(t) \leq e^{-t} \text{ for all } n, \ t > 0 .$$

Thus by the dominated convergence theorem:

$$\lim_{n\to\infty} \rho_n = 1-\lim_{n\to\infty} \int_0^\infty \overline{H}_n(t)dt = 1 - \int_0^\infty [1-\{(1-e^{-ct})/c\}]dt$$

$$= \begin{cases} \infty & \text{for } c < 1 \\ 0 & \text{for } c = 1 \\ -\infty & \text{for } c > 1 . \end{cases}$$

But NBUE distributions satisfy $0 \leq \rho \leq 1/2$ (since $\mu_2 \geq \mu^2$ by Chebychev's inequality). Thus c equals 1 and $\lim \rho_n = 0$.

4. <u>Potential Improvement of NBUE Bound</u>. In the following example we have a sequence of IFRA (and thus NBUE) distributions $\{F_n, n \geq 1\}$, with

$$\lim_{n\to\infty} \frac{\sup|\overline{F}_n(x)-e^{-x/\mu_n}|}{\rho_n^{1/2}} = 1 .$$

It follows from this example and theorem 1 that the best bound of the form $c\rho^{\alpha}$ has $\alpha = 1/2$ and $1 \leq c \leq \frac{4\sqrt{6}}{\pi}$. Thus the maximum potential improvement in the bound $A\rho^{1/2}$ is the lowering of A to 1. This statement holds for the NBUE class as well as for the subclasses NBU and IFRA.

The distribution F_n is defined by:

$$\overline{F}_n(t) = \begin{cases} 1 & t < \frac{1}{n} \\ e^{-t} & t \geq \frac{1}{n} . \end{cases}$$

Then:

$$\mu_n = n^{-1} + e^{-n^{-1}}$$

$$\mu_{2,n} = n^{-2} + 2(n^{-1}+1)e^{-n^{-1}}$$

$$\rho_n = 1-[(1+2n(n+1)e^{-n^{-1}})/2(1+ne^{-n^{-1}})^2]$$

$$D_n = \sup|\overline{F}_n(x)-e^{-x/\mu_n}| = 1-\exp[-1/(1+ne^{-n^{-1}})] .$$

It follows that:

$$D_n = n^{-1} + o(n^{-1})$$

and

$$\rho_n = n^{-2} + o(n^{-2}) \ .$$

Thus:

$$\lim_{n\to\infty} [D_n/|\rho_n|^{1/2}] = 1 \ .$$

5. NWUE Results. Assume that F is NWUE. Applying Lemma 2.2 with $F_1 = F$ and $F_2 = G$ we obtain:

$$\sup|\overline{F}(x)-\overline{G}(x)| \le A\rho^{1/2} \ . \tag{5.1}$$

We do not know of an analogue of (3.2) for NWUE distributions, but applying Lemma 2.2 with $\overline{F}_1(x) = e^{-x/\mu}$ and $F_2 = G$ we obtain:

$$\sup|\overline{G}(x)-e^{-x/\mu}| \le A\rho^{1/2} \ . \tag{5.2}$$

Since $\overline{G}(x) \ge \max(\overline{F}(x), e^{-x/\mu})$ it follows from (5.1) and (5.2) that:

$$\sup|\overline{F}(x)-e^{-x/\mu}| \le A\rho^{1/2} \ . \tag{5.3}$$

Finally since $e^{-x/\mu} \le \min(\overline{G}(t), e^{-t/\mu_G})$ and $\sup|e^{-t/\mu_G}-e^{-t/\mu}| \le 1-(\mu/\mu_G) = \rho/\rho+1$, we obtain:

$$\sup|\overline{G}(x)-e^{-x/\mu_G}| \le A\rho^{1/2} \ . \tag{5.4}$$

Corollary (3.7) does not hold for NWUE distributions. While (5.3) insures that $\lim \rho_n = 0$ is sufficient for convergence to an exponential distribution, $\lim \rho_n = 0$ is not a necessary condition. To see this consider the distribution F with failure rate:

$$h(x) = \begin{cases} 2 & 0 \le x \le 1 \\ 2x^{-1} & x > 1 . \end{cases}$$

Clearly F is DFR, with finite mean, and infinite second moment. Now, for $n = 1,2,\ldots$ define:

$$F_n(x) = n^{-1}F(x) + (1-n^{-1})(1-e^{-x}) .$$

Since F_n is a mixture of DFR distributions, F_n is DFR and thus NWUE. Clearly, F_n converges to an exponential distribution with mean 1. However since F has infinite second moment, so does F_n, and thus $\rho_n = \infty$ for all n.

6. Geometric Sums. Y is defined to be a geometric sum of X with parameter p if Y can be represented as $\Sigma_1^N X_i$ with $\{X_i, i \ge 1\}$ i.i.d. as X, N geometrically distributed with parameter p, and N and $\{X_i\}$ independent.

Lemma 6.1. If Y is a geometric sum of X with parameter p then $\rho_Y = p\rho_X$ where $\rho_Y = |(EY^2/2(EY)^2)-1|$ and $\rho_X = |(EX^2/2(EX)^2)-1|$.

Proof. Define $\mu = EX$, $\mu_2 = EX^2$, $\sigma^2 = \text{Var}\,X$, $\mu_Y = EY$, $\mu_{2,Y} = EY^2$ and $\sigma_Y^2 = \text{Var}\,Y$. Note that:

$$\mu_Y = \mu/p \tag{6.2}$$

$$\sigma_Y^2 = q\mu^2 p^{-2} + \sigma^2 p^{-1} \ . \tag{6.3}$$

Thus:

$$\mu_{2,Y} = (1+q)\mu^2 p^{-2} + \sigma^2 p^{-1} \ . \tag{6.4}$$

From (6.2) and (6.4):

$$\mu_{2,Y}/2\mu_Y^2 = q + (\mu_2 p/2\mu^2) \ . \tag{6.5}$$

We see from (6.5) that $\mu_{2,Y}/2\mu_Y^2 \leq 1$ if and only if $\mu_2/2\mu^2 \leq 1$. Assume that $\mu_2/2\mu^2 \leq 1$. Then:

$$\rho_Y = 1-(\mu_{2,Y}/2\mu_Y^2) = p[1-(\mu_2/2\mu^2)] = p\rho_X \ . \tag{6.6}$$

Finally for $\mu_2/2\mu^2 \leq 1$:

$$\rho_Y = (\mu_{2,Y}/2\mu_Y^2)-1) = p[(\mu_2/2\mu^2)-1] = p\rho_X \tag{6.7}$$

and the result is proved.

Lemma 6.8. Suppose that Y is a geometric sum of X where X is NBUE. Then Y is NBUE. The analogous result holds for X NWUE.

Proof. Consider a renewal process with interarrival time distribution X. Binomial sampling with probability p of the renewal epochs leads to an embedded renewal process with interarrival time distribution Y. It follows that Y^* can be represented as $X^* + \Sigma_1^{N-1} X_i$, where $X^*(Y^*)$ has the stationary distribution of $X(Y)$. $\{X_i\}$ is i.i.d. as X, N is geometric with parameter p, and X^*, $\{X_i\}$, and N are independent. But, Y is representable as $X + \Sigma_1^{N-1} X_i$ with X, N and $\{X_i\}$ independent. Now if X is NBUE then X is stochastically greater than X^* so $Y = X + \Sigma_1^{N-1} X_i$ is stochastically greater than $Y^* = X^* + \Sigma_1^{N-1} X_i$, and Y is thus NBUE. The analogous argument obviously works for X NWUE.

Theorem 6.9. Let X be either NBUE or NWUE with finite second moment. Suppose that Y is a geometric sum of X with parameter p. Then:

$$\sup|\Pr(Y > t) - e^{-tp\mu^{-1}}| \leq A(p\rho)^{1/2}$$

where $A = \frac{4\sqrt{6}}{\pi}$ and $\rho = |(EX^2/2(EX)^2)-1|$.

Proof. The result follows from Theorem 3.6, Lemma 6.7 and Lemma 6.8.

Note that Theorem 6.9 applies to defective renewal processes, which are discussed in Feller [3], chapter XI, sections 6 and 7.

References

[1] Brown, M. (1981). "Approximating IMRL distributions by exponential distributions with applications to first passage times." To appear in Ann. Probability.

[2] Brown, M. (1983). "Inequalities for distributions with increasing failure rate." City College, CUNY Report No. MB-3. To appear in March 1983.

[3] Feller, W. (1966). An Introduction to Probability Theory and Its Applications, Volume II. John Wiley, New York.

[4] Hall, P. (1979). "On measures of the distance of a mixture from its parent." Stochastic Process. Appl. 8, 357-65.

[5] Heyde, C.C. (1975). "Kurtosis and departures from normality." Statistical Distributions in Scientific Work, Vol. I, edited by Patil, Katz, and J.K. Ord. Reidel, Dortrecht, 193-220.

[6] Heyde, C.C. and Leslie, J.R. (1976). "On moment measures of departure from normal and exponential laws." Stochastic Process. Appl. 4, 317-28.

[7] Keilson, J. (1979). Markov Chain Models: Rarity and Exponentiality. Springer-Verlag, N.Y.

END

FILMED

6-83

DTIC